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THE COMPLEXITY OF CELLS IN THREE-DIMENSIONAL ARRANGEMENTS*

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A set of m planes dissects E^3 into cells, facets, edges and vertices. Letting $\deg(c)$ be the number of facets that bound a cell c, we give exact and asymptotic bounds on the maximum of $\sum_{c \text{ in } C} \deg(c)$, if C is a family of cells of the arrangement with fixed cardinality.

1. Introduction

A finite set H of planes in three-dimensional Euclidean space E^3 induces a cell complex called the *arrangement* $A(H)$ of H. While the two-dimensional analogue[1] has received a great deal of attention in the mathematical literature (see [12] for an excellent collection of results obtained until 1972) less is known about combinatorial properties of arrangements in three dimensions. We refer to [10] and [11] for discussions of arrangements in E^3 and in higher dimensions[2].

In E^3, $A(H)$ consists of four kinds of faces called *vertices, edges, facets,* and *cells.* Upper bounds on the number of faces in an arrangement are well-known (see [3], [10], [1], etc.): if $|H| = n$ then $A(H)$ consists of at most $\binom{n}{3}$ vertices, $3\binom{n}{3} + \binom{n}{2}$ edges, $3\binom{n}{3} + 2\binom{n}{2} + n$ facets, and $\binom{n}{3} + \binom{n}{2} + n + 1$ cells. These bounds are tight if and only if $A(H)$ is *simple,* that is, if any three planes of H intersect in a point, and no four do so.

Let the *degree* of a cell c, denoted $\deg(c)$, be the number of facets incident with c. For a collection C of cells in $A(H)$, we define $a_C(H) = \sum_{c \text{ in } C} \deg(c)$. Thus each facet is counted once for each cell it bounds. Then $a_k(H) = \max\{a_C(H) \mid |C| = k\}$, and finally $a_k(n) = \max\{a_k(H) \mid |H| = n\}$. This paper is devoted to giving upper and lower bounds on $a_k(n)$; the same problem in E^2 it tackled in [13], [4], [8]. By Euler's theorem for polytopes in three dimensions,[3]

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[1] An arrangements in E^2 is a dissection induced by a finite set of lines.

[2] In general, a finite set of hyperplanes in E^d defines an arrangement.

[3] Euler's theorem reads: $V - E + F = 2$, with V, E, and F the number of vertices, edges, and facets of c.

each cell c is bounded by at most $2\deg(c)-4$ vertices and at most $3\deg(c)-6$ edges. Thus $2a_k(n)-4k$ and $3a_k(n)-6k$ give the maximum number of vertices and edges of a collection of k cells in an arrangement of n planes (again, provided an edge or vertex is counted once for each cell it bounds).

Part of the motivation for considering bounds on $a_k(n)$ stems from several problems in computational geometry that require storing parts of arrangements. Examples are the construction of transversals for line segments ([6]) and order-k Voronoi diagrams ([7]). If k cells of an n-plane arrangement are stored, then the maximal amount of storage needed is proportional to $a_k(n)$.

From a mathematical point of view it is interesting to note that all results given in this paper also hold for arrangements of pseudoplanes. (A pseudoplane is a surface in E^3 homeomorphic to E^2, and the arrangement of any three of them is isomorphic to a simple arrangement of three planes.) Therefore, $a_k(n)$ also reflects properties of rank-3 oriented matroids ([2]), and, by correspondence to arrangements of planes, of configurations of points in E^3 and zonotopes in E^4 ([5]).

The organization of this paper is as follows: Section 2 gives exact bounds on $a_k(n)$ for very small and very large k, Sections 3 and 4 present asymptotic lower and upper bounds for general k, and finally Section 5 discusses the results obtained and gives open problems.

2. Exact bounds for Extreme k

We start with small collections of cells. Throughout, n and k are assumed such that $a_k(n)$ is well-defined.

Fact 2.1. $a_1(n)=n$.

Fact 2.1 says that a single cell c is bounded by at most n facets, which is immediate since the convexity of c forbids a plane to support more than one of c's facets.

Corollary 2.2. $a_k(n)\leq kn$.

The upper bound of Corollary 2.2 is tight for very small k.

Theorem 2.3. *$a_k(n)=kn$, for $k\leq 5$.*

Proof. The assertion is verified in two steps: We first construct a set H of five planes with $a_5(H)=25$. This proves the theorem for $n\leq 5$. Then we demonstrate the existence of five cells in $A(H)$ such that:

(i) each has degree five, and
(ii) for every $n>5$, there are $t=n-5$ planes that can be added to H such that the degree of each of these cells is increased to n.

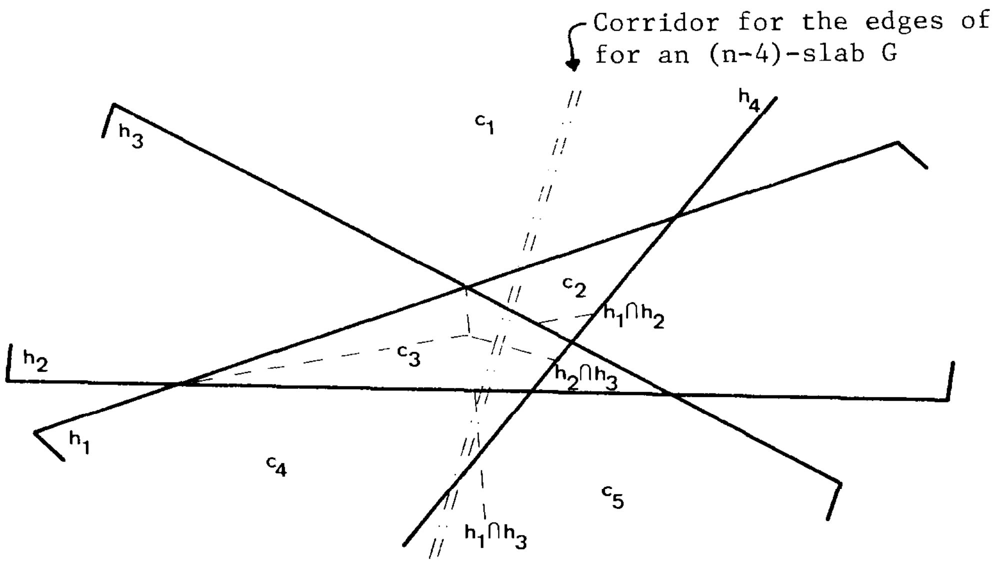

Fig. 1. 5-plane arrangement.

Choose $H = \{h_0, h_1, h_2, h_3, h_4\}$ such that h_0 is the xy-plane and h_1 to h_4 intersect h_0 as shown in Fig. 1. Next choose the angles between h_0 and h_i $(i = 1, \ldots, 4)$ such that each plane touches each of cells c_1, c_2, c_3, c_4, and c_5 below h_0 as depicted, that is, each plane supports a facet of each c_1 to c_5: $\deg(c_3) = 5$ in any case, and slanting h_1, h_2 and h_3 as indicated guarantees $\deg(c_i) = 5$, for $i = 1, \ldots, 5$.

To extend $A(H)$, we call an arrangement $A(G)$ a $|G|$-*slab* if all planes in G are normal to a common plane $h(G)$, and there is an unbounded cell $c(G)$ in $A(G)$ with $\deg(c(G)) = |G|$. Figure 2 depicts a 7-slab with the planes normal to the drawing plane. There are $|G| - 1$ (unbounded), edges in the boundary of $c(G)$, and for every positive real ε and positive integer m there is an m-slab G with any two edges at most ε units of length apart. Furthermore, this can be achieved for

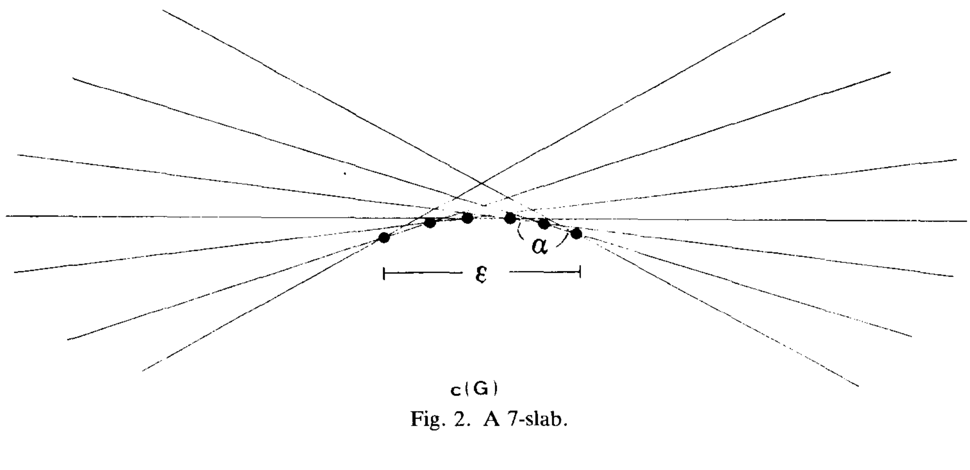

Fig. 2. A 7-slab.

the minimal angle at an edge in $c(G)$ at least α, for every $\alpha < 180°$ (see Fig. 2). We replace h_0 in H by an $(n-4)$-slab G such that:

(i) the edges of $c(G)$ fit into a corridor immediately below h_0 (as indicated in Fig. 1);
(ii) $c(G)$ is below h_0, and
(iii) $c(G)$ intersects each facet of c_1 to c_5 (except for those supported by h_0).

Taking c_i' as the intersection of c_i and $c(G)$ yields $\deg(c_i') = n$, for $i = 1, \ldots, 5$. □

The authors have not been able to calculate the exact value of $a_6(n)$, but they venture

Conjecture 2.4. *$a_6(n) < 6n$, for n sufficiently large.*

Now we turn our attention to extremely large collections of cells, that is, k equal to or insignificantly smaller than $\binom{n}{3} + \binom{n}{2} + n + 1$, the maximal number of cells in an arrangement of n planes.

Fact 2.5. If $k = \binom{n}{3} + \binom{n}{2} + n + 1$, then $a_k(n) = 2(3\binom{n}{3} + 2\binom{n}{2} + n)$.

This is obvious, since when every cell is in C, each facet is counted exactly twice. We can extend this result as follows. Let $t(H)$ be the number of cells c in $A(H)$ with $\deg(c) = 3$, and define $t(n) = \max\{t(H) \mid |H| = n \text{ and } A(H) \text{ simple}\}$. Then we note

Fact 2.6. $a_k(n) = 3\binom{n}{3} + \binom{n}{2} - n - 3 + 3k$, for $k \geq \binom{n}{3} + \binom{n}{2} + n + 1 - t(n)$.

Fact 2.6 follows immediately from Fact 2.5 by subtracting $3(\binom{n}{3} + \binom{n}{2} + n + 1 - k)$ facets, i.e., three facets for each cell of degree 3 that can be omitted.

3. Asymptotic lower bounds

The first result of this section assures that the upper bound of Corollary 2.2 has an at least asymptotically matching lower bound for $k = O(n)$.[4]

Theorem 3.1. *$a_k(n) = \Omega(kn)$, for $k = O(n)$.*

Proof. Let $m_1 = \lfloor \frac{1}{2}n \rfloor$ and let $A(G_1)$ be an m_1-slab as defined in Section 2, with planes in G_1 normal to the plane $h(G_1)$. Choose G_2 to contain $m_2 = n - m_1$ planes

[4] A function $g(n)$ is $O(f(n))$ if there exist $c > 0$ and n_0 such that $g(n) \leq cf(n)$ whenever $n > n_0$. $g(n)$ is $\Omega(f(n))$ if $f(n)$ is $O(g(n))$. $g(n)$ is $\Theta(f(n))$ if it is both $O(f(n))$ and $\Omega(f(n))$.

all parallel to $h(G_1)$ and thus normal to the planes in G_1. Obviously, $A(G_1 \cup G_2)$ contains $m_2 - 1$ cells with degree $m_1 + 2$ each. Taking k of these cells proves the assertion for $k \leqslant m_2 - 1$. Augmenting the $m_2 - 1$ cells with any $k - m_2 + 1$ other cells establishes the assertion for $k > m_2 - 1$. $\square$

We immediately derive also

Corollary 3.2. $a_k(n) = \Omega(n^2)$, *for* $k = \Omega(n)$.

This lower bound is the best result obtained by the authors for $k = O(n^{3/2})$. In fact they venture

Conjecture 3.3. $a_k(n) = O(n^2)$, *for* $k = O(n^{3/2})$.

Next, we give a non-trivial lower bound by extending a construction of [8].

Theorem 3.4. $a_k(n) = \Omega(k^{2/3}n)$.

Proof. Ignoring uninteresting cases, we can assume $12n \leqslant k \leqslant n^3/64$, $n \geqslant 8$, and define $m = \lfloor \frac{1}{4}n \rfloor$ and $l = \lfloor (k/12n)^{1/2} \rfloor < \frac{1}{4}m$. Now let $P = \{(a, b, c) \mid 1 \leqslant a \leqslant m$ and $1 \leqslant b, c \leqslant l\}$, a set of points in E^3 with integer coordinates. For any plane h in E^3, call $|P \cap h|$ the *contribution* of h, denoted contr(h). We show that there is a set H of at most $2m$ planes such that:

(i) each point p in P is a vertex in $A(H)$, and

(ii) $\sum_{h \text{ in } H} \mathrm{contr}(h) = \Omega(l^{4/3}m^{5/3})$.

This result implies the assertion by the following argument:

(1) Replace each point p of P by a ball $b(p)$ with center p and radius ε.

(2) Replace each plane h in H by two planes h' and h'' parallel to h such that h' and h'' touch $b(p)$ on different sides if h contains p.

(3) Choosing ε small enough, a point p in P contained in i planes gives rise to a cell containing $b(p)$ with degree $2i$. Hence, summing up the degrees of $b(p)$, for all p in P yields $\Omega(l^{4/3}m^{5/3}) = \Omega(k^{2/3}n)$.

We continue with the construction of H. To make the points of P vertices of $A(H)$, $G_1 = \{x = a, y = b, z = c \mid 1 \leqslant a \leqslant m, 1 \leqslant b, c \leqslant l\} \subseteq H$. $|G_1| = m + 2l$. Let $h(i, j, r, s)$ be the plane (parallel to the x-axis) passing through $(1, i, j)$, (m, i, j), and $(1, i+r, j+s)$. To complete H we let $H = G_1 \cup G_2$ with $G_2 = \{h(i, j, r, s) \mid 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant \lfloor \frac{1}{2}l \rfloor,\ 1 \leqslant r \leqslant c_0(m/l)^{1/3}$, where c_0 is a suitable positive constant to be specified later, and $1 \leqslant s \leqslant r$, where r, s are relatively prime$\}$. All planes in G_2 are distinct, and

$$|G_2| = \lfloor \tfrac{1}{2}l \rfloor \sum_{r=1}^{c_0(m/l)^{1/3}} r\Phi(r), \text{ with } \Phi(r) = |\{s\colon 1 \leqslant s \leqslant r \text{ and } r, s \text{ are relatively prime}\}|.$$

Since $\sum_{r=1}^{N} r\Phi(r) = \Theta(N^3)$ (see [9]), we derive $|G_2| = \Theta(m)$ while $|G_2| \leq m - 2l$, for suitable c_0. Notice that our calculation is not incorrect if $r > l$; in this case the corresponding planes are not defined, the calculated contribution of each such plane is no more than m, and it is easy to find substituting planes with at least this contribution. Hence $|H| \leq 2m$.

The contribution of $h(i, j, r, s)$ is at least $m(l/2r)$. The overall contribution of G_2 is therefore at least

$$\lfloor \tfrac{1}{2}l \rfloor \sum_{t=1}^{c_0(m/l)^{1/3}} \left(m \frac{l}{2r}\right) r\Phi(r) \geq m \lfloor \tfrac{1}{2}l \rfloor^2 \sum_{r=1}^{c_0(m/l)^{1/3}} \Phi(r) = \Theta(l^{4/3} m^{5/3}),$$

since $\sum_{r=1}^{N} \Phi(r) = \Theta(N^2)$([9]). $\square$

4. Asymptotic upper bounds

Note that by Corollary 2.2 we have

Fact 4.1. $a_k(n) = O(kn)$.

By Theorem 3.1, this upper bound is asymptotically tight for $k = O(n)$. For $k = \Omega(n)$, a better upper bound can be derived from the following result of [7].

Lemma 4.2. *Let H be a set of n hyperplanes in E^d, and $C(H)$ be the set of cells in $A(H)$. Then* $\sum_{c \text{ in } C(H)} \deg^2(c) = \Theta(n^d)$.

In particular for $d = 3$, Lemma 4.2 asserts $\sum_{c \text{ in } C(H)} \deg^2(c) = \Theta(n^3)$.

Theorem 4.3. $a_n(n) = O(k^{1/2} n^{3/2})$.

Proof. Let C be a collection of cells in $A(H)$ with $|H| = n$, $|C| = k$ and $a_C(H) = a_k(n)$. By Lemma 4.2, $\sum_{c \text{ in } C} \deg^2(c) \leq \sum_{c \text{ in } C(H)} \deg^2(c) = \Theta(n^3)$. To maximize $\sum_{c \text{ in } C} \deg(c)$, this constraint forces all cells to have about equal degree, that is, $\deg(c) = \Theta((n^3/k)^{1/2})$, for all c in C. The assertion follows immediately. $\square$

Defining $a_k^{(d)}(n)$ to denote the maximal number of $(d-1)$-faces in an arrangement of n hyperplanes in E^d, Theorem 4.3 can obviously be generalized to

$$a_k^{(d)}(n) = O(k^{1/2} n^{d/2}).$$

For $d = 2$, this bound is demonstrated in [8] using a different argument.

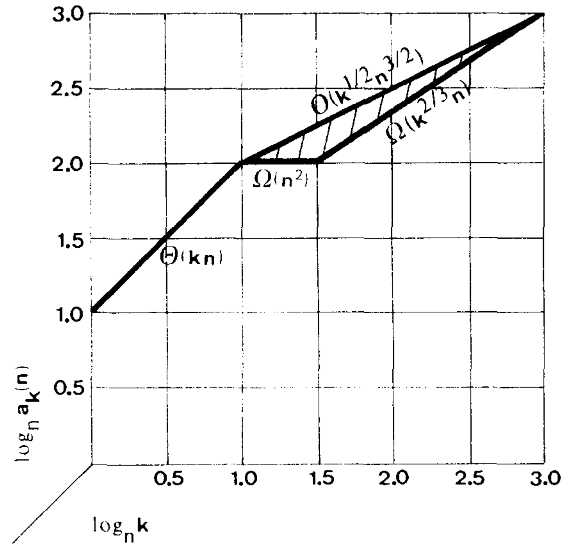

Fig. 3. Asymptotic results on $a_k(n)$.

5. Discussion

Exact and asymptotic upper and lower bounds on the maximal number $a_k(n)$ of facets that bound k cells in an arrangement of n planes in E^3 are demonstrated. Figure 3 displays the asymptotic results using the logarithms base n of k and of $a_k(n)$. The shaded area makes the gap in our asymptotic results obvious. The authors are inclined to believe that the lower bounds of Corollary 3.2 and Theorem 3.4 are tight. In particular, they invite the reader to prove or disprove Conjecture 3.3.

To extend the exact results of Theorem 2.3 and Fact 2.6 is another challenge. To this end a resolution of Conjecture 2.4 and bounds on $t(n)$ (defined in Section 2) would be an important first step. The analysis of 7-plane arrangements given in [14] may be of some help here, although this work is restricted to projective space.

References

[1] G.L. Alexanderson and J.E. Wetzel, Simple partitions of space, Math. Mag. 51 (1978) 220–225.
[2] R.G. Bland and M. Las Vergas, Orientability of matroids, J. Combin. Theory Ser. B 24 (1978) 94–123.
[3] R.C. Buck, Partition of space, Amer. Math. Monthly 50 (1943) 541–544.
[4] R.J. Canham, A theorem on arrangements of lines in the plane, Israel J. Math. 7 (1969) 393–397.
[5] H.S.M. Coxeter, Regular Polytopes (Dover, New York, 1973).
[6] H. Edelsbrunner, H.A. Maurer, F.P. Prepartata, A.L. Rosenberg, E. Welzl and D. Wood, Stabbing line segments, BIT 22 (1982) 274–281.

[7] H. Edelsbrunner, J. O'Rouke and R. Seidel, Constructing arrangements of lines and hyperplanes with applications, Rep. F123, Inst. Inf. Proc., TU Graz, Austria (1983).
[8] H. Edelsbrunner and E. Welzl, On the maximal number of edges of many faces in an arrangement, J. Combin Theory Ser. A, to appear.
[9] M.L. Fredman, The Inherent Complexity of Dynamic Data Structures which Accommodate Range Queries, Proc. 21st Ann. Symp. Found. Comp. Sci. (1980) 191–199.
[10] B. Grünbaum, Convex Polytopes (Interscience, London, 1967).
[11] B. Grünbaum, Arrangements of hyperplanes, Congr. Num. III, Louisiana Conf. Comb., Graph Th., Comp. (1971) 41–106.
[12] B. Grünbaum, Arrangements and Spreads, Reg. Conf. Series in Math. (Amer. Math. Soc., Providence, 1972).
[13] F. Levi, Die Teilung der projektiven Ebene durch Gerade und Pseudogerade, Ber. Math.-Phys. Kl. Saechs. Akad. Wiss. Leipzig 78 (1926) 256–267.
[14] H.S. White, The Convex Cells formed by 7-Planes, Proc. Nat. Acad. Sci. 25 (1939) 147–153.